Cooperative learning in a congestion game

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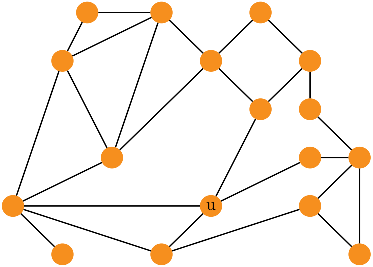
Multi-agent networked systems

The classical scenario: many simple units dynamically interacting through a network.

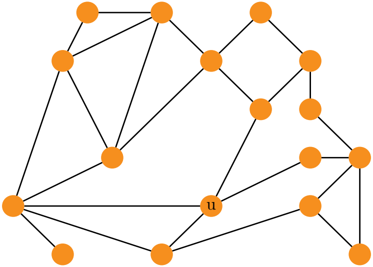


Examples:

- Social networks. Units are people exchanging opinions.
- Economic/Financial networks. Units are people, companies, banks with business relations.
- Biological networks

Multi-agent networked systems

The classical scenario: many simple units dynamically interacting through a network.



Key issues:

- Equilibrium: Analysis when time $\to \infty$.
- Large Scale: Analysis when number of units $N \to \infty$.
- Resilience Dynamical effects of local perturbations
- Phase transitions Emergence of global phenomena

Multi-agent networked systems

In many applications, the assumption that units only interact through a network is not appropriate.

In socio-economic or financial networks, units (people, companies) may be influenced by other agents to which they are directly connected (friends, partners, ecc), but they are also implicitly connected through media, global markets.

If I buy a device (or choose a specific resource) following the suggestion of a friend, my action influences the future price of the device (resource) and so, possibly, the future choices of other people

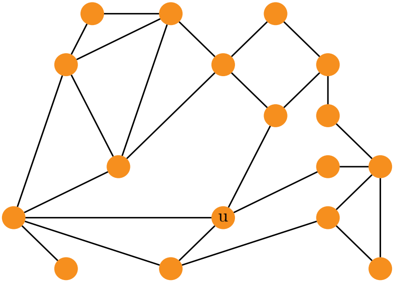
Need to consider models where local and global interactions are simultaneously present.

Outline

- Review of classical interacting particle models (voter model)
- Interaction model with a global congestion term
- Hydrodynamic limit: Nash equilibrium is asymptotically reached
- Conclusions and future research lines

Interacting particle models

$G = (V, E)$ connected graph



$x_u(t) \in \mathcal{X}$ state of unit u at time t. In this talk $\mathcal{X} = \{0, 1\}$.

Randomly a unit u activates at time t and changes state:

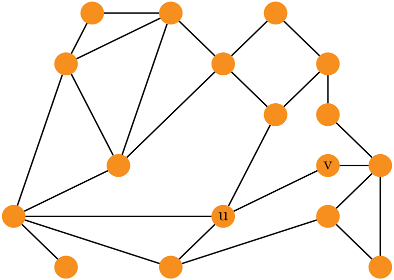
$$x_u(t) \longrightarrow x_u(t^+) = f_u(x_v(t) \mid v \text{ neighbor of } u)$$

f_u deterministic or stochastic.

Markov process on the configuration space $\mathcal{X}^V$.

Interacting particle models

$G = (V, E)$ connected graph



Examples for the interacting mechanism:

- spontaneous autonomous flipping
- gossip interaction (voter model): u chooses a neighbor v at random and copies its state.
- f_u maximizes some local utility function (Game theory)

Mean field models

Full communication (graph is complete)
Homogeneous units ($f_u = f$ independent on u)

$z(t) =$ fraction of units in state 1 at time t $\longrightarrow$ Markov process

$q^+(z)$ ($q^-(z)$) probability that the number of 1's will increase (decrease) of one unit given that the fraction of 1's is z.

Theorem (Hydrodynamic limit $N \to +\infty$)

Under mild regularity assumptions on q^+ and q^-

- *For every $T > 0$, the process on $[0, T]$ $z(t)$ almost surely converges to a solution of the ODE*
$$z' = q^+(z) - q^-(z)$$
- *If the Markov process is ergodic and π is the unique invariant probability,*
$$\mathrm{wk}\lim_{N\to+\infty} \pi = \sum \lambda_j \delta_{z_j}$$
where z_j's are the stable equilibrium points of the ODE.

Mean field models

Example: voter model with spontaneous flipping:

q_g probability of a gossip step
q_f probability of a flipping step

$$q^+(z) = q_g(1-z)z + q_f(1-z)\,, \quad q^-(z) = q_g z(1-z) + q_f z$$

$$z' = (1-2z)q_f$$

For $q_f \neq 0$, $1/2$ is an asymptotically stable equilibrium

Weak convergence of equilibrium measure $\pi \to \delta_{1/2}$

Other large scale models

Limit for $N \to +\infty$ difficult in general

- grid-like graphs
- graphs with specific degree distributions
- small world

$z(t)$ is not Markovian!

π equilibrium on $\mathcal{X}^V$ is a too large object.

Thermodynamic approach: global observable $\mu : \mathcal{X}^V \to \mathbb{R}$

Analysis of π_μ distribution of μ

- μ=fraction of 1's, $\pi_\mu = \pi(z)$
- μ= fraction of disagreement links $(0, 1)$

An interacting model with a congestion term

At every time t, units receive a reward which depends on their own state and on the fraction of population sharing the same state

$$R_u(t) = U(x_u(t), z(t)) + \omega_u(t)$$

Congestion: $U(0, z)$ increasing in z; $U(1, z)$ decreasing in z.
$\omega_u(t)$ Gaussian i.i.d. noises

Assumption $U(0, z) = U(1, z)$ has exactly one solution $z_{\mathrm{Nash}} \in (0, 1)$: if $z = z_{\mathrm{Nash}}$, the mean rewards for a unit in 0 or in 1 are equal.

Flipping + Gossip interaction mechanism: with probability q_f a flipping happens; with probability q_g a gossip interaction happens: a link (u, v) is activated and u copies the state of v if $R_u(t) < R_v(t)$. The result of the gossip interaction depends on the global variable z!

Mean field analysis

$$q^+(z) = q_f(1-z)+q_g(1-z)z\theta(z)\,, \quad q^-(z) = q_f z+q_g z(1-z)(1-\theta(z))$$

where

$$\theta(z) = \mathbb{P}(R(0,z)+\omega' < R(1,z)+\omega'') = \mathbb{P}(\omega > R(0,z)-R(1,z))$$

$$\omega', \omega'' \sim N(0,\sigma^2); \quad \omega \sim N(0,2\sigma^2).$$

$$z' = q_f(1-2z) + q_g(1-z)z[2\theta(z)-1]$$

Equilibria (for $q_f = 0$): $0,\ 1,\ z_{\mathrm{Nash}}$
$(\theta(z) = 1/2 \Leftrightarrow R(0,z) = R(1,z))$

Equilibria (for q_f small): $z_0,\ z_1,\ \bar{z}$

$$\mathrm{wk} \lim_{q_f \to 0} \lim_{N \to +\infty} \pi = \delta_{z_{\mathrm{Nash}}}$$

Mean field analysis is easy but not very interesting: local/global dichotomy is lost!

Main result

Theorem

For families of expander graphs

$$\mathrm{wk}\lim_{q_f \to 0} \lim_{N \to +\infty} \pi = \delta_{z_{\mathrm{Nash}}}$$

Expander graphs include:

- random Erdos-Renyi graphs
- random graphs with assigned degree distribution
- Barabasi preferential attachment graphs

The theorem shows that through the gossip interactions, a learning process is taking place: in spite of the noisy rewards (variance is fixed!), the population reaches the Nash equilibrium.

Sketch of proof

$z(t)$ no longer Markovian.

However, if $\eta(t) =$ fraction of disagreement links at time t.

$\mathbb{P}(z(t+) = z + 1/N \mid z(t) = z, \eta(t) = \eta) = q_f(1-z) + \frac{q_g}{2}\eta\theta(z)$

$\mathbb{P}(z(t+) = z - 1/N \mid z(t) = z, \eta(t) = \eta) = q_f z + \frac{q_g}{2}\eta(1-\theta(z))$

This leads to a balanced equation on $\pi(z)$:

$$q_f[\pi(z) - z^+\pi(z^+) - z^-\pi(z^-)] + \frac{q_g}{2}[\eta_\pi(z) - (1-\theta(z^+))\eta_\pi(z^+) - \theta(z^-)\eta_\pi(z^-)]$$

where $z^+ = z + 1/N$, $z^- = z - 1/N$

$\eta_\pi(z) = \mathbb{E}_{\pi(z)}[\eta]$ (mean value of η conditioned to be in z).

Sketch of proof

Hydrodynamic limit: integrate over a C^1 function $g : [0,1] \to \mathbb{R}$ and take the limit:

the limit points π, η_π are measures on $[0,1]$ s.t.

$$\int_0^1 \left[q_f(1-2z) + \frac{q_g}{2}(2\theta(z)-1)\frac{\mathrm{d}\eta_\pi}{\mathrm{d}\pi}(z) \right] g'(z)\mathrm{d}\pi(z) = 0$$

This implies that π must be supported where

$$q_f(1-2z) + \frac{q_g}{2}(2\theta(z)-1)\frac{\mathrm{d}\eta_\pi}{\mathrm{d}\pi}(z) = 0$$

Sketch of proof

Analysis of the equation:

$$q_f(1-2z) + \frac{q_g}{2}(2\theta(z)-1)\frac{\mathrm{d}\eta_\pi}{\mathrm{d}\pi}(z) = 0$$

Complete case: $\frac{\mathrm{d}\eta_\pi}{\mathrm{d}\pi}(z) = 2z(1-z)$. For $q_f \to 0$, $\pi \to \delta_{z_{\mathrm{Nash}}}$

For expander graphs $\frac{\mathrm{d}\eta_\pi}{\mathrm{d}\pi}(z) \geq \delta z(1-z)$ and the same result is obtained.

This completes the proof

Conclusions

We have proposed an interacting particle model where units are engaged in a congestion game and compare experiences through gossip interaction. Our main result shows that in expander graphs, when the number of units is large and the flipping noise small, they dynamically reach the Nash equilibrium.

Many open issues:

- Analyze the behavior on graphs which are not expanders
- Analyze the fraction of active links η at equilibrium. Which configurations are provileged?
- Study different models where local and global influences are present.